

# Existence of homoclinic solution in first order discrete Hamiltonian system

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## Abstract

In this paper we consider the first order discrete Hamiltonian system

$$\begin{cases} x_1(n+1) - x_1(n) &= -H_{x_2}(n, x(n)), \\ x_2(n) - x_2(n-1) &= H_{x_1}(n, x(n)). \end{cases}$$

Where  $n \in \mathbb{Z}$ ,  $x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \in \mathbb{R}^{2N}$ ,  $H(n, z) = \frac{1}{2}S(n)z \cdot z + R(n, z)$  is periodic in  $n$  and asymptotically quadratic as  $|z| \rightarrow \infty$ . We will prove the existence of homoclinic solution by critical point theorem for strongly indefinite functional.

**Keywords:** discrete Hamiltonian system, homoclinic solution, asymptotically quadratic, variational method

## 1. Introduction and main results

In this paper we are interested in the following discrete first order Hamiltonian system

$$\begin{cases} x_1(n+1) - x_1(n) &= -H_{x_2}(n, x(n)), \\ x_2(n) - x_2(n-1) &= H_{x_1}(n, x(n)). \end{cases} \quad (DHS)$$

Where  $n \in \mathbb{Z}$  and  $x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \in \mathbb{R}^{2N}$ ,  $H(n, \cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R})$  depends periodically on  $n$  and has the form

$$H(n, z) = \frac{1}{2}S(n)z \cdot z + R(n, z)$$

with  $S(n)$  being a symmetric  $2N \times 2N$  real matrix. Let

$$Lx(n) = \begin{pmatrix} x_1(n+1) \\ x_2(n) \end{pmatrix}, \quad \Delta x(n) = x(n+1) - x(n),$$

and

$$\mathcal{J} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix},$$

then we can rewrite system (DHS) as follows

$$\Delta Lx(n-1) = \mathcal{J} \nabla H(n, x(n)) \quad n \in \mathbb{Z}. \quad (DHS)'$$

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We are interested in the existence of homoclinic solution  $x = (x(n))_{n \in \mathbb{Z}}$  of  $(DHS)$ , i.e.  $x \neq 0$  and  $x(n) \rightarrow 0$  as  $|n| \rightarrow \infty$ .

System  $(DHS)$  can be regarded as a discrete analog of continuous Hamiltonian system

$$\begin{cases} \dot{x}_1(t) &= -H_{x_2}(t, x(t)), \\ \dot{x}_2(t) &= H_{x_1}(t, x(t)). \end{cases} \quad (CHS)$$

which have been largely studied in the literature of the existence and multiplicity of homoclinic orbits by different approaches. Especially, there are some significant results for  $(CHS)$  via variational method. For details, we refer to [2-13] and references therein.

In last years, there have been many studies on discrete Hamiltonian systems by different approaches. Abounding researches have been made on boundary value problems, oscillations and asymptotic behavior of discrete Hamiltonian systems (see for example [20-24]). By critical point theory, the existence and multiplicity of periodic solutions have been considered in [15-18]. It's well known Hamiltonian systems are very important in the study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. While it is well known that homoclinic solutions play an important role in analyzing the chaos of Hamiltonian systems. If a system has the transversely intersected homoclinic solutions, then it must be chaotic. If it has the smoothly connected homoclinic solutions, then it cannot stand the perturbation, its perturbed system probably produces chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic solutions of Hamiltonian systems. As we know, there are not much results for homoclinics of discrete Hamiltonian systems. As in [19], the existence of homoclinic solutions has been obtained in second order discrete Hamiltonian systems. Recently, Chen, Yang and Ding [14] proves existence and multiplicity of homoclinics in first order Hamiltonian systems with the nonlinear term being super-quadratic. Hence our aim of this paper is to establish some existence results for first order discrete Hamiltonian system with asymptotically quadratic term.

We will discuss our results variationally. Inspired by [11], we discuss the associated linear self-adjoint operator  $A + S$  (defined in Section 2). By the spectrum of  $A + S$ , we establish the variational framework for  $(DHS)$ . And we show that the associated functional  $\Phi$  is strongly indefinite.

To state our results, we use the notation  $\mathcal{J}_0 = \begin{pmatrix} 0 & -I_N \\ -I_N & 0 \end{pmatrix}$ . For a sequence of symmetric matrixs  $\{L(n)\}$ , let  $\mathfrak{e}(n)$  be the set of all eigenvalues of  $L(n)$  and set

$$\lambda_L = \inf_n \mathfrak{e}(n), \quad \Lambda_L = \sup_n \mathfrak{e}(n).$$

In particular, we set  $\lambda_0 := \lambda_{\mathcal{J}_0 S}$  and  $\Lambda_0 := \Lambda_{\mathcal{J}_0 S}$ . And we will use the notation

$$\tilde{R}(n, z) = \frac{1}{2} \nabla R(n, z) z - R(n, z)$$

We make the following hypotheses:

$(R_0)$  There is a positive integer  $T$  such that  $S(n + T) = S(n)$ , and  $\mathcal{J}_0 S(n)$  is symmetric and positive definite for all  $n \in \mathbb{Z}$ .

$(R_1)$   $R(n + T, z) = R(n, z)$ ,  $\forall n \in \mathbb{Z} \forall z \in \mathbb{R}^{2N}$ ,  $R(n, \cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ .

(R<sub>2</sub>)  $R(n, z) \geq 0$ ,  $\nabla R(n, z) = o(|z|)$  as  $|z| \rightarrow 0$ .

(R<sub>3</sub>)  $\nabla R(n, z) - S_\infty(n)z = o(|z|)$  as  $|z| \rightarrow \infty$ , where  $S_\infty(n)$  is a symmetric matrix with  $\lambda_\infty := \lambda_{S_\infty} > 2 + \Lambda_0$

(R<sub>4</sub>)  $\tilde{R}(n, z) \geq 0$  and there is  $\delta_0 \in (0, \lambda_0)$  such that if  $|\nabla R(n, z)| \geq (\lambda_0 - \delta_0)|z|$  then  $\tilde{R}(n, z) \geq \delta_0$ .

Then we have the main result of this paper.

**Theorem 1.1** Let (R<sub>0</sub>)–(R<sub>4</sub>) be satisfied. Then the system (DHS) has at least one homoclinic solution.

This paper is organized as follows. In Section 2, we establish the variational framework of the problem and recall some abstract critical point theories on strongly indefinite functional. In Section 3, we discuss the linking structure of  $\Phi$  and the behavior of  $(C)_c$ -sequence. Finally, in Section 4, we prove our results.

## 2. Variational setting

Let  $E := l^2(\mathbb{Z}, \mathbb{R}^{2N})$ .  $E$  is a Hilbert space with the usual inner product and norm

$$(x, y)_E = \sum_n x(n) \cdot y(n) \quad |x|_E^2 = \sum_n |x(n)|^2 \quad x, y \in E \quad (2.1)$$

On  $E$  we define functional  $\Phi$ , for any  $x \in E$ ,

$$\Phi(x) = -\frac{1}{2} \sum_n \mathcal{J} \Delta L x(n-1) \cdot x(n) - \frac{1}{2} \sum_n S(n) x(n) \cdot x(n) - \sum_n R(n, x(n)). \quad (2.2)$$

For convenience, we define operators as follows.

$$A : E \rightarrow E : \quad Ax = (z(n))_{n \in \mathbb{Z}} \quad z(n) = -\mathcal{J} \Delta L x(n-1), \quad x \in E \quad (2.3)$$

$$S : E \rightarrow E : \quad Sx = (z(n))_{n \in \mathbb{Z}} \quad z(n) = -S(n)x(n), \quad x \in E. \quad (2.4)$$

Then  $A$  and  $S$  are linear bounded self-adjoint operators(see[14] ).

Moreover, we set

$$\Psi(x) := \sum_n R(n, x(n)) \quad (2.5)$$

Thus, we can rewrite functional  $\Phi$ :

$$\Phi(x) = \frac{1}{2}((A + S)x, x)_E - \Psi(x). \quad (2.6)$$

Since Hilbert space  $E = l^2$  embeds continuously into  $l^p(\mathbb{Z}, \mathbb{R}^{2N})(2 < p \leq \infty)$ , i.e.

$$|x|_{l^p} \leq |x|_E \quad x \in E, \quad (2.7)$$

and (R<sub>1</sub>) – (R<sub>3</sub>) imply that , for any  $\varepsilon > 0, p \geq 2$ , there is  $C_\varepsilon > 0$  such that

$$|\nabla R(n, z)| \leq \varepsilon |z| + C_\varepsilon |z|^{p-1} \quad (2.8)$$

and

$$|R(n, z)| \leq \varepsilon |z|^2 + C_\varepsilon |z|^p. \quad (2.9)$$

Hence the functional  $\Phi$  is well defined. By a standard argument, one can obtain that  $\Phi$  is  $C^1$  and has Fréchet derivative of the following form

$$\Phi'(x)y = - \sum_n \mathcal{J} \Delta L x(n-1) \cdot y(n) - \sum_n S(n)x(n) \cdot y(n) - \sum_n \nabla R(n, x(n)) \cdot y(n) \quad (2.10)$$

for  $x, y \in E$ .

Obviously, if  $x \in E$  is a critical point of  $\Phi$  then  $x$  is a solution of (1), moreover,  $x(n) \rightarrow 0$  as  $|n| \rightarrow \infty$ . Since  $S(n)$  and  $R(n, z)$  are  $T$ -periodic on  $n$ , it is not difficult to see that  $\Phi$  is  $T$ -periodic.

In order to establish a variational setting for the system (*DHS*) we study the spectrum of the associated linear self-adjoint operator  $A + S$ . Let  $\sigma(A + S)$ ,  $\sigma_e(A + S)$  denote, respectively, the spectrum and essential spectrum of  $A + S$ . Supposing  $(R_0)$  holds, by the definition of  $\lambda_0$ , it is easy to see that  $\lambda_0 > 0$ .

**Proposition 2.1** Assume  $(R_0)$  is satisfied. Then

$$1^\circ \quad \sigma(A + S) = \sigma_e(A + S);$$

$$2^\circ \quad \sigma(A + S) \cap (0, \infty) \neq \emptyset \text{ and } \sigma(A + S) \cap (-\infty, 0) \neq \emptyset;$$

$$3^\circ \quad \sigma(A + S) \subset [-\Lambda_0 - 2, -\lambda_0] \cup [\lambda_0, \Lambda_0 + 2].$$

**Proof.** The proofs of  $1^\circ, 2^\circ$  and  $\sigma(A + S) \subset \mathbb{R} \setminus (-\lambda_0, \lambda_0)$ , one can obtain in [14]. To prove that  $3^\circ$  holds, it is sufficient to show that  $\|A + S\| \leq 2 + \Lambda_0$ . Note that

$$\begin{aligned} |Ax|_{\ell^2}^2 &= (Ax, Ax)_{\ell^2} = \sum_n (-\mathcal{J} \Delta L x(n-1)) \cdot (-\mathcal{J} \Delta L x(n-1)) \\ &= \sum_n \{|x_2(n)|^2 + |x_2(n-1)|^2 + |x_1(n)|^2 + |x_1(n+1)|^2 \\ &\quad - 2x_2(n) \cdot x_2(n-1) - 2x_1(n) \cdot x_1(n+1)\} \\ &\leq \sum_n \{2|x_1(n)|^2 + 2|x_1(n+1)|^2 + 2|x_2(n)|^2 + 2|x_2(n-1)|^2\} \\ &= 4|x|_{\ell^2}^2 \end{aligned}$$

Observe that  $\mathcal{J}_0^2 = I$  and  $\mathcal{J}_0 S(n) = S(n)\mathcal{J}_0$ , we have

$$\begin{aligned} |Sx|_{\ell^2}^2 &= (Sx, Sx)_{\ell^2} = \sum_n (-S(n)x(n)) \cdot (-S(n)x(n)) = \sum_n \mathcal{J}_0^2 S(n)x(n) \cdot \mathcal{J}_0^2 S(n)x(n) \\ &= \sum_n \mathcal{J}_0 S(n) \mathcal{J}_0 x(n) \cdot \mathcal{J}_0 S(n) \mathcal{J}_0 x(n) = \sum_n (\mathcal{J}_0 S(n))^2 \mathcal{J}_0 x(n) \cdot \mathcal{J}_0 x(n). \end{aligned}$$

By definition of  $\Lambda_0$ , we have that  $|Sx|_{\ell^2}^2 \leq \Lambda_0^2 |x|_{\ell^2}^2$ . It follows that  $\|A\| \leq 2$  and  $\|S\| \leq \Lambda_0$ . Therefore,  $\|A + S\| \leq \|A\| + \|S\| \leq 2 + \Lambda_0$ . The proof is complete.

Due to the spectrum of  $A + S$ ,  $E = l^2(\mathbb{Z}, \mathbb{R}^{2N})$  possesses the orthogonal decomposition

$$E = E^- \oplus E^+, \quad x = x^- + x^+, \quad (2.11)$$

corresponding to the spectrum decomposition of  $A + S$  such that

$$((A + S)x, x)_\rho \leq -\lambda_0|x|_\rho^2 \text{ on } E^- \text{ and } ((A + S)x, x)_\rho \geq \lambda_0|x|_\rho^2 \text{ on } E^+. \quad (2.12)$$

Let  $|A + S|$  denote the absolute value of  $A + S$ , we equip  $E$  with the inner product

$$(x, y) = (|A + S|^{1/2}x, |A + S|^{1/2}y)_\rho.$$

Then  $(E, (\cdot, \cdot))$  is a Hilbert space, and it has the associated norm  $\|x\| = (x, x)^{1/2}$ . By 3° of Proposition 2.1, one can obtain that

$$\lambda_0|x|_\rho \leq \|x\| \leq (2 + \Lambda_0)|x|_\rho, \quad (2.13)$$

which implies that  $(E, \|\cdot\|)$  is equivalent to  $(E, |\cdot|_\rho)$ . It is not difficult to see that the decomposition of  $E$  is orthogonal with respect to both  $(\cdot, \cdot)_\rho$  and  $(\cdot, \cdot)$ . Now, we can rewrite the functional  $\Phi$  as

$$\Phi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \Psi(x). \quad (2.14)$$

Hence, by the Proposition 2.1,  $\Phi$  is strongly indefinite.

To study the critical point of  $\Phi$ , we recall some abstract critical point theory developed in [1].

Let  $Z$  be a Banach space with direct sum decomposition  $Z = X \oplus Y$  and corresponding projections  $P_X, P_Y$  onto  $X, Y$ , respectively. For a functional  $\Phi \in C^1(Z, \mathbb{R})$  we write  $\Phi_a = \{z \in Z : \Phi(z) \geq a\}$ ,  $\Phi^b = \{z \in Z : \Phi(z) \leq b\}$  and  $\Phi_a^b = \Phi_a \cap \Phi^b$ . Recall that  $\Phi$  is said to be weakly sequentially lower semicontinuous if for any  $z_n \rightarrow z$  in  $Z$  one has  $\Phi(z) \leq \liminf_{n \rightarrow \infty} \Phi(z_n)$ , and  $\Phi'$  is said to be weakly sequentially continuous if  $\lim_{n \rightarrow \infty} \Phi'(z_n)w = \Phi'(z)w$  for each  $w \in Z$ . A sequence  $(z_n) \in Z$  is said to be a  $(C)_c$ -sequence if  $\Phi(z_n) \rightarrow c$  and  $\Phi'(z_n) \rightarrow 0$ .  $\Phi$  is said to satisfy the  $(C)_c$ -condition if any  $(C)_c$ -sequence has a convergent subsequence.

From now on, let  $X$  be separable and reflexive, and fix a countable dense subset  $\mathcal{S} \subset X^*$ . For each  $s \in \mathcal{S}$  there is a semi-norm on  $Z$  defined by

$$p_s : Z \rightarrow \mathbb{R}, p_s(z) = |s(x)| + \|y\| \quad \text{for } z = x + y \in Z = X \oplus Y.$$

We denote by  $\mathcal{T}_\mathcal{S}$  the induced topology, Let  $w^*$  denote the *weak\**-topology on  $Z^*$ .

Suppose:

( $\Phi_0$ ) For any  $c \in \mathbb{R}$ ,  $\Phi_c$  is  $\mathcal{T}_\mathcal{S}$ -closed, and  $\Phi' : (\Phi_c, \mathcal{T}_\mathcal{S}) \rightarrow (E^*, w^*)$  is continuous.

( $\Phi_1$ ) For any  $c > 0$ , there exists  $\zeta > 0$  such that  $\|z\| \leq \zeta\|P_Y z\|$  for all  $z \in \Phi_c$ .

( $\Phi_2$ ) There exists  $\rho > 0$  with  $\kappa = \inf \Phi(S_\rho Y) > 0$  where  $S_\rho Y = \{y \in Y : \|y\| = \rho\}$ .

Then the following theorem is a special case of Theorem 4.4 of [1].

**Theorem 2.2** Let  $(\Phi_0) - (\Phi_2)$  be satisfied and assume there is  $R > \rho$  and  $e \in Y$  with  $\|e\| = 1$  such that  $\sup \Phi(\partial Q) \leq \kappa$  where  $Q = \{z = x + te : t \geq 0, x \in X, \|z\| < R\}$ . Then there is a  $(C)_c$ -sequence for  $\Phi$  with  $c \in [\kappa, \sup \Phi(Q)]$ .

To check that functional  $\Phi$  satisfies  $(\Phi_0)$ , the following proposition (cf. [1]) is a key tool.

**Proposition 2.3** Suppose  $\Phi \in C^1(Z, \mathbb{R})$  is of the form

$$\Phi(z) = \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 - \Psi(z) \quad z = x + y \in Z = X \oplus Y \quad (2.15)$$

such that

- (i)  $\Psi \in C^1(Z, \mathbb{R})$  is bounded from below.
- (ii)  $\Psi$  is weakly sequentially lower semicontinuous
- (iii)  $\Psi'$  is weakly sequentially continuous.
- (iv)  $\nu : Z \rightarrow \mathbb{R}, \nu(z) = \|z\|^2$ , is  $C^1$  and  $\nu' : (Z, \mathcal{T}_w) \rightarrow (Z^*, \mathcal{T}_{w^*})$  is sequentially continuous.

Then  $\Phi$  in (2.15) satisfies  $(\Phi_0)$ .

### 3. Linking structure and $(C)_c$ sequence

In order to apply Theorem 2.1, we study the linking structure of  $\Phi$ .

**Lemma 3.1** Let  $(R_0) - (R_3)$  be satisfied. Then there exists  $\rho > 0$  such that  $\kappa = \inf \Phi(S_\rho^+) > 0$  where  $S_\rho^+ = \partial B_\rho \cap E^+$ .

*Proof* Choose  $p > 2$  such that (2.9) holds for any  $\varepsilon > 0$ . We get that

$$\Psi(x) \leq \varepsilon \|x\|_{\rho^2}^2 + C_\varepsilon \|x\|_{l^p}^p \leq (2 + \Lambda_0)(\varepsilon \|x\|^2 + C_\varepsilon \|x\|^p)$$

for all  $x \in E$ . We choose  $\varepsilon$  small enough, then the lemma holds from the form of  $\Phi$ .

**Lemma 3.2** Let  $(R_0) - (R_3)$  be satisfied. Then for any  $e \in E^+$  with  $\|e\| = 1$  there exists  $R > 0$  such that  $\Phi(x) \leq \kappa$  for all  $x \in E^- \oplus \mathbb{R}e$ , with  $\|x\| \geq R$ .

*Proof* It is sufficient to show that  $\Phi(x) \rightarrow -\infty$  as  $x \in E^- \oplus \mathbb{R}e, \|x\| \rightarrow \infty$ . Arguing by indirectly, assume that for some sequence  $x_k = s_k e + x_k^- \in E^- \oplus \mathbb{R}e$  with  $\|x_k\| \rightarrow \infty$ , there is  $M > 0$  such that  $\Phi(x_k) \geq -M$  for all  $k$ . Then, setting  $y_k = x_k / \|x_k\| := t_k e + y_k^-$ , we have  $\|y_k\| = 1$ ,  $y_k \rightharpoonup y$ ,  $y_k^- \rightharpoonup y^-$ ,  $t_k \rightarrow t \in \mathbb{R}$  and

$$-\frac{M}{\|x_k\|^2} \leq \frac{1}{2}|t_k|^2 - \frac{1}{2}\|y_k^-\|^2 - \frac{\sum_n R(n, x_k(n))}{\|x_k\|^2}. \quad (3.1)$$

Remark that  $t \neq 0$ . Indeed, if not then it follows from (3.1) that

$$0 \leq \frac{1}{2}\|y_k^-\|^2 + \frac{\sum_n R(n, x_k(n))}{\|x_k\|^2} \leq \frac{1}{2}|t_k|^2 + \frac{M}{\|x_k\|^2}, \quad (3.2)$$

in particular,  $\|y_k^-\| \rightarrow 0$ , hence  $1 = \|y_k\| \rightarrow 0$ , a contradiction.

Since  $(R_3)$ , there holds

$$\begin{aligned} |t|^2 - \|y^-\|^2 - \sum_n S_\infty(n) y(n) \cdot y(n) &\leq \|te\|^2 - \|y^-\|^2 - \lambda_\infty \|y\|_{\rho^2}^2 \\ &\leq (2 + \Lambda_0)|t|^2 |e|_{\rho^2}^2 - \|y^-\|^2 - \lambda_\infty |t|^2 |e|_{\rho^2}^2 - \lambda_\infty \|y^-\|_{\rho^2}^2 \\ &= (2 + \Lambda_0 - \lambda_\infty)|t|^2 |e|_{\rho^2}^2 - \|y^-\|^2 - \lambda_\infty \|y^-\|_{\rho^2}^2 < 0. \end{aligned}$$

Hence for some  $\tilde{N} > 0$

$$|t|^2 - \|y^-\|^2 - \sum_{|n| \leq \tilde{N}} S_\infty(n) y(n) \cdot y(n) < 0. \quad (3.3)$$

Let  $G(n, z) := R(n, z) - \frac{1}{2} S_\infty(n) z \cdot z$ , by  $(R_3)$  and (2.9), one gets that  $|G(n, z)| \leq C_2 |z|^2$ .

Since

$$\begin{aligned} \lim_{k \rightarrow \infty} \left( \sum_{|n| \leq \tilde{N}} \frac{R(n, x_k(n))}{\|x_k\|^2} - \frac{1}{2} \sum_{|n| \leq \tilde{N}} S_\infty(n) y_k(n) \cdot y_k(n) \right) &= \lim_{k \rightarrow \infty} \sum_{|n| \leq \tilde{N}} \frac{G(n, x_k(n))}{\|x_k\|^2} \\ &= \lim_{k \rightarrow \infty} \sum_{|n| \leq \tilde{N}} \frac{G(n, x_k(n)) |y_k(n)|^2}{|x_k(n)|^2}, \end{aligned}$$

and we have that

$$\left| \sum_{|n| \leq \tilde{N}} \frac{G(n, x_k(n)) |y_k(n)|^2}{|x_k(n)|^2} \right| \leq \sum_{|n| \leq \tilde{N}} \frac{|G(n, x_k(n))| |y_k(n)|^2}{|x_k(n)|^2}.$$

For  $|n| \leq \tilde{N}$ , if  $y_k(n) \rightarrow 0$  then  $\frac{|G(n, x_k(n))| |y_k(n)|^2}{|x_k(n)|^2} \leq C_2 |y_k(n)|^2 \rightarrow 0$ , otherwise, if  $y_k(n) \not\rightarrow 0$ , then  $|x_k(n)| \rightarrow \infty$ , which yields that  $\frac{|G(n, x_k(n))| |y_k(n)|^2}{|x_k(n)|^2} \leq C_3 \frac{|G(n, x_k(n))|}{|x_k(n)|^2} \rightarrow 0$  ( $C_3 > 0$ ). Thus,

$$\sum_{|n| \leq \tilde{N}} \frac{G(n, x_k(n)) |y_k(n)|^2}{|x_k(n)|^2} \rightarrow 0.$$

It follows that

$$0 \leq \lim_{k \rightarrow \infty} \left( \frac{1}{2} |t_k|^2 - \frac{1}{2} \|y_k^-\|^2 - \sum_{|n| \leq \tilde{N}} \frac{R(n, x_k(n))}{\|x_k\|^2} \right) \leq \frac{1}{2} (|t|^2 - \|y^-\|^2 - \sum_{|n| \leq \tilde{N}} S_\infty(n) y(n) \cdot y(n)) < 0,$$

that is a contradiction.

It follows from Lemma 3.1 and Lemma 3.2 that  $\Phi$  has linking structure which is showed by following lemma.

**Lemma 3.3** Under the assumptions of Lemma 3.2, letting  $e \in E^+$  with  $\|e\| = 1$ , there is  $R_0 > \rho$  such that  $\sup(\partial Q) \leq \kappa$  where  $Q := \{x = x^- + te : t \geq 0, x^- \in E^-, \|x\| < R_0\}$ .

Now we discuss the behavior of  $(C)_c$ -sequence.

**Lemma 3.4** Let  $(R_0) - (R_4)$  be satisfied, then any  $(C)_c$ -sequence of  $\Phi$  is bounded.

*Proof* Let  $(x_k) \subset E$  be such that

$$\Phi(x_k) \rightarrow c \quad (1 + \|x_k\|)\Phi'(x_k) \rightarrow 0. \quad (3.4)$$

Then, for some  $C_0 > 0$ ,

$$C_0 \geq \Phi(x_k) - \frac{1}{2} \Phi'(x_k) x_k = \sum_n \tilde{R}(n, x_k(n)). \quad (3.5)$$

Arguing by indirectly assume up to a subsequence  $\|x_k\| \rightarrow \infty$ . Set  $y_k = x_k / \|x_k\|$ . Then,  $\|y_k\| = 1$ . Remark that

$$\Phi'(x_k)(x_k^+ - x_k^-) = \|x_k\|^2 \left( 1 - \frac{\sum_n \nabla R(n, u_k(n)) (y_k^+(n) - y_k^-(n))}{\|x_k\|} \right)$$

it follows from (3.4) that

$$\frac{\sum_n \nabla R(n, u_k(n)) (y_k^+(n) - y_k^-(n))}{\|x_k\|} \rightarrow 1 \quad (3.6)$$

If  $\exists \tau > 0$ ,  $\exists n_k \in \mathbb{Z}$  such that  $|y_k(n_k)| \geq \tau$ . There exist  $l_k \in \mathbb{Z}$  such that  $0 \leq n_k - l_k T \leq T - 1$ . Set  $\tilde{x}_k = \{\tilde{x}_k(n)\}$  where  $\tilde{x}_k(n) = x_k(n + l_k T)$  and  $\tilde{y}_k = \{\tilde{y}_k(n)\}$  where  $\tilde{y}_k(n) = y_k(n + l_k T)$ . For any  $w \in E$  setting  $\tilde{w}_k = \{\tilde{w}_k(n)\}$  where  $\tilde{w}_k(n) = w_k(n - l_k T)$ , and define operator  $S_\infty$  as

$$S_\infty : E \rightarrow E : \quad S_\infty x = (z(n))_{n \in \mathbb{Z}} \quad z(n) = S_\infty(n)x(n), \quad x \in E.$$

We have that

$$\begin{aligned} \Phi'(x_k)\tilde{w}_k &= (x_k^+ - x_k^-, \tilde{w}_k) - (S_\infty x_k, \tilde{w}_k)_{l^2} - \sum_n \nabla R(n, x_k(n))\tilde{w}_k(n) \\ &= \|x_k\| \left( (y_k^+ - y_k^-, \tilde{w}_k) - (S_\infty y_k, \tilde{w}_k)_{l^2} - \sum_n \nabla R(n, x_k(n))\tilde{w}_k(n) \frac{|y_k(n)|}{|x_k(n)|} \right) \end{aligned}$$

By  $A + S$ ,  $S_\infty$  and  $\nabla R(n, z)$  are periodic in  $n$ , then

$$\Phi'(x_k)w_k = \|x_k\| \left( (\tilde{y}_k^+ - \tilde{y}_k^-, w) - (S_\infty \tilde{y}_k, w)_{l^2} - \sum_n \nabla R(n, \tilde{x}_k(n))w(n) \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \right)$$

This follows

$$(\tilde{y}_k^+ - \tilde{y}_k^-, w) - (S_\infty \tilde{y}_k, w)_{l^2} - \sum_n \nabla R(n, \tilde{x}_k(n))w(n) \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \rightarrow 0$$

Since  $\|\tilde{y}_k\| = \|y_k\| = 1$ , then up to subsequence we obtain that  $\tilde{y}_k \rightharpoonup \tilde{y}$ ,  $\tilde{y}_k(n) \rightarrow \tilde{y}(n)$ . For some  $n_0 : 0 \leq n_0 \leq (T - 1)$  such that  $|\tilde{y}_k(n_0)| \geq \tau$ , hence,  $\tilde{y} \neq 0$ . By (2.8), one gets that  $|\nabla R(n, z)| \leq C_4|z|$ . For  $w \in E = l^2$  and  $\varepsilon > 0$ , there exists  $\tilde{N} \in \mathbb{N}$  such that  $\sum_{|n| > \tilde{N}} |w(n)|^2 < \varepsilon^2$ . Note that

$$\begin{aligned} & \left| \sum_n \nabla R(n, \tilde{x}_k(n))w(n) \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \right| \leq \sum_n |R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \\ & \leq \sum_{|n| \leq \tilde{N}} |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} + \sum_{|n| > \tilde{N}} |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \\ & \leq \sum_{|n| \leq \tilde{N}} |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} + C_4|y_k|_{l^2} \left( \sum_{|n| > \tilde{N}} |w(n)|^2 \right)^{\frac{1}{2}} \\ & \leq \sum_{|n| \leq \tilde{N}} |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} + C_5\varepsilon. \end{aligned}$$

Similarly to the proof of Lemma 3.2, we can obtain that

$$\sum_{|n| \leq \tilde{N}} |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \rightarrow 0.$$

It follows that

$$\sum_n |\nabla R(n, \tilde{x}_k(n))||w(n)| \frac{|\tilde{y}_k(n)|}{|\tilde{x}_k(n)|} \rightarrow 0,$$

hence

$$(\tilde{y}^+ - \tilde{y}^-, w) - (S_\infty \tilde{y}, w)_{l^2} = 0.$$



Thus, 0 is an eigenvalue of the operator  $A + S - S_\infty$  with eigenfunction  $\tilde{y}$ . We claim that is impossible. Indeed, by the definition of  $\lambda_\infty$  and (2.13), one has that, for any  $x \in E$

$$|(A + S)x - S_\infty x|_{l^2} \geq |S_\infty x|_{l^2} - |(A + S)x|_{l^2} \geq (\lambda_\infty - 2 - \Lambda_0)|x|_{l^2}.$$

It follows from  $(R_3)$  that  $\lambda_\infty - 2 - \Lambda_0 > 0$  which yields that  $0 \notin \sigma(A + S - S_\infty)$ .

If for any  $\tau > 0$  and  $n$  there holds  $|y_k(n)| < \tau$ , then  $|y_k|_{l^\infty} \rightarrow 0$ . Since  $|y_k|_{l^p}^p \leq |y_k|_{l^\infty}^{p-2}|y_k|_{l^2}^2$  ( $p > 2$ ) and  $y_k$  is bounded in  $E = l^2$ , we get that  $|y_k|_{l^p} \rightarrow 0$  ( $p > 2$ ). In virtue of  $(R_4)$ , we set

$$I_k := \left\{ n \in \mathbb{Z} : \frac{|\nabla R(n, x_k(n))|}{|x_k(n)|} \leq \lambda_0 - \delta_0 \right\}$$

Since  $\lambda_0|y_k|_{l^2}^2 \leq \|y_k\|^2 = 1$ , we have

$$\begin{aligned} \left| \sum_{I_k} \frac{\nabla R(n, x_k(n))(y_k^+(n) - y_k^-(n))}{\|x_k\|} \right| &= \left| \sum_{I_k} \frac{\nabla R(n, x_k(n))(y_k^+(n) - y_k^-(n))|y_k(n)|}{|x_k(n)|} \right| \\ &\leq (\lambda_0 - \delta_0)|y_k|_{l^2}^2 \leq \frac{\lambda_0 - \delta_0}{\lambda_0} < 1 \end{aligned}$$

for all  $k$ . Let  $I_k^c = \mathbb{Z} \setminus I_k$ , jointly with (3.6), implies that

$$\lim_{k \rightarrow \infty} \left| \sum_{I_k^c} \frac{\nabla R(n, x_k(n))(y_k^+(n) - y_k^-(n))}{\|x_k\|} \right| > 1 - \frac{\lambda_0 - \delta_0}{\lambda_0} = \frac{\delta_0}{\lambda_0}.$$

Recalling (2.8), we can choose  $C > 0$  such that  $|\nabla R(n, z)| \leq C|z|$ , there holds for an arbitrarily fixed  $s > 2$ ,

$$\begin{aligned} \left| \sum_{I_k^c} \frac{\nabla R(n, x_k(n))(y_k^+(n) - y_k^-(n))}{\|x_k\|} \right| &\leq C \sum_{I_k^c} |y_k^+(n) - y_k^-(n)||y_k(n)| \leq C|y_k|_{l^2}|I_k^c|^{(s-2)/s}|y_k|_{l^s} \\ &\leq \frac{C}{\lambda_0}|I_k^c|^{(s-2)/s}|y_k|_{l^s} \end{aligned}$$

Since  $|y_k|_{l^s} \rightarrow 0$ , one gets that  $|I_k^c| \rightarrow \infty$ . By  $(R_4)$ ,  $\tilde{R}(n, x_k(n)) \geq \delta_0$  on  $I_k^c$ , hence

$$\sum_n \tilde{R}(n, x_k(n)) \geq \sum_{I_k^c} \tilde{R}(n, x_k(n)) \geq |I_k^c|\delta_0 \rightarrow \infty$$

contrary to (3.5). The proof is finished.

#### 4. Proof of main result

We are now in a position to give the proof of our main result. In order to apply the abstract Theorem 2.2, we choose  $X = E^-$  and  $Y = E^+$  with  $E^\pm$  given in Section 2.  $X$  is separable and reflexive and let  $\mathcal{S}$  be a countable dense subset of  $X^*$ . First we have

**Lemma 4.1**  $\Phi$  satisfies  $(\Phi_0)$  and  $(\Phi_1)$ .

*Proof.* In virtue of the form of  $\Phi$  and Proposition 2.3, to show that  $\Phi$  satisfies  $(\Phi_0)$  it is sufficient to show that  $\Psi$  is bounded from below,  $\Psi$  is weakly sequentially lower semicontinuous and  $\Psi'$  is weakly sequentially continuous.

Firstly, since  $R(n, z)$  is non-negative, so is  $\Psi$ . Secondly, let  $x_k \rightharpoonup x$  in  $E$ . We have that  $x_k(n) \rightarrow x(n)$  as  $k \rightarrow \infty$ , hence,  $R(n, x_k(n)) \rightarrow R(n, x(n))$ . Thus,

$$\Psi(x) = \sum_n \lim_{k \rightarrow \infty} R(n, x_k(n)) \leq \liminf_{k \rightarrow \infty} \sum_n R(n, x_k(n)) = \liminf_{k \rightarrow \infty} \Phi(x_k)$$

which implies that  $\Psi$  is weakly sequentially lower semicontinuous.

Thirdly, let  $x_k \rightharpoonup x$  in  $E$ . By (2.8), we choose  $C_1 > 0$  such that  $|\nabla R(n, z)| \leq C_1|z|$ . For any  $y \in E$ , one can get that for any  $\varepsilon > 0$  there is  $N \in \mathbb{N}$  such that  $\sum_{|n| \geq M} |y(n)|^2 < \varepsilon$ . By Hölder inequality,

$$\begin{aligned} |\Psi'(x_k)y - \Psi'(x)y| &\leq \left| \sum_{|n| > N} (\nabla R(n, x_k(n)) - \nabla R(n, x(n))) \cdot y(n) \right| \\ &\quad + \left| \sum_{|n| \leq N} (\nabla R(n, x_k(n)) - \nabla R(n, x(n))) \cdot y(n) \right| \\ &\leq C_1 \varepsilon (|x_k|_{l^2} + |x|_{l^2}) + \left| \sum_{|n| \leq N} (\nabla R(n, x_k(n)) - \nabla R(n, x(n))) \cdot y(n) \right|. \end{aligned}$$

Observe that  $\nabla R(n, x_k(n)) \rightarrow \nabla R(n, x(n))$  for each  $n \in \mathbb{Z}$ . It follows from  $|n| \leq N$  is finite that  $\left| \sum_{|n| \leq N} (\nabla R(n, x_k(n)) - \nabla R(n, x(n))) \cdot y(n) \right| \rightarrow 0$ . Therefore we obtain that  $\Psi'$  is weakly sequentially continuous.

For any  $c > 0$ , since  $\Psi$  is non-negative, it is not difficult to see that  $\|x^-\| \leq \|x^+\|$ . Hence one can get that  $\|x\| \leq 2\|x^+\|$  for  $x \in \Phi_c$  which shows that  $(\Phi_1)$  holds. The proof is complete.

**Proof of theorem 1.1** Lemma 4.1 implies that  $\Phi$  satisfies  $(\Phi_0)$  and  $(\Phi_1)$ . And Lemma 3.3 shows that  $\Phi$  has linking structure. Hence there is a  $(C)_c$ -sequence  $(x_k)$  with level  $c \geq \kappa > 0$ . Lemma 3.4 shows that  $(x_k)$  is bounded:  $\|x_k\| \leq M$ . In addition,

$$c = \lim_{k \rightarrow \infty} \left( \Phi(x_k) - \frac{1}{2} \Phi'(x_k)x_k \right) = \lim_{k \rightarrow \infty} \sum_n \tilde{R}(n, x_k(n)) \quad (4.1)$$

Remark that there are  $\tau > 0$  and  $n_k \in \mathbb{Z}$  such that  $|x_k(n_k)| \geq \tau$ . Indeed, if not, it follows from the proof of Lemma 3.3 that  $|x_k|_{l^p} \rightarrow 0$  ( $p > 2$ ). By (2.8) and (2.9), choose  $p > 2$ , such that for any  $\varepsilon > 0$ , there is  $C_\varepsilon > 0$  satisfying  $\tilde{R}(n, z) \leq \varepsilon \lambda_0^2 M^{-2} |z|^2 + C_\varepsilon |z|^p$ , then it follows from (4.1) that, for  $\varepsilon < c$ ,

$$c = \lim_{k \rightarrow \infty} \sum_n \tilde{R}(n, x_k(n)) \leq \lim_{k \rightarrow \infty} \left( \varepsilon \lambda_0^2 M^{-2} |x_k|_{l^2}^2 + |x_k|_{l^p}^p \right) \leq \varepsilon,$$

a contradiction. Make proper shifts similar to the proof of Lemma 3.4, then passing to a subsequence  $\tilde{x}_k$  such that there is  $\tau > 0$  and  $0 \leq n_0 \leq (T - 1)$  independent of  $k$ ,  $|\tilde{x}_k(n_0)| \geq \tau$ . Due to periodicity of the coefficients,  $\tilde{x}_k$  is also a Cerami-sequence for  $\Phi$  at  $c$ . Hence  $\tilde{x}_k \rightharpoonup \tilde{x} \neq 0$ , thus, we obtain a nontrivial critical point  $\tilde{x}$  of  $\Phi$ .

## REFERENCES

1. T. Bartsch, Y. H. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory, Math. Nachr. 279 (2006) 1267-1288.
2. T. Bartsch, Y. H. Ding, Homoclinic solutions of an infinite-dimensional Hamiltonian system, Math. Z. 240 (2002) 289-310

3. G. Arioli and A. Szulkin, Homoclinic solutions of Hamiltonian system with symmetry, *J. Differential Equations* 158 (1999) 291-313.
4. V. Coti-Zelati, I. Ekeland and E. Séré, A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann.* 288 (1990) 133-160.
5. Y. H. Ding, M. Girardi, Infinitely many homoclinic orbits of a Hamiltonian system with symmetry, *Nonlinear Anal.* 38 (1999) 391-415.
6. Y. H. Ding, M. Willem, Homoclinic orbits of a Hamiltonian system, *Z. Angew. Math. Phys.* 50 (1999) 759-778.
7. H. Hofer and K. Wysocki, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, *Math. Ann.* 288 (1990) 483-503.
8. E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.* 209 (1992) 27-42.
9. A. Szulkin, W. Zou, Homoclinic orbits for asymptotically linear Hamiltonian systems, *J. Funct. Anal.* 187 (2001) 25-41.
10. K. Tanaka, Homoclinic orbits in first order superquadratic Hamiltonian system: Convergence of subharmonic orbits, *J. Differential Equations* 94 (1991) 315-339.
11. Y. H. Ding, Multiple Homoclinic in a Hamiltonian System with asymptotically or super linear terms, *Comm. Cont. Math.* 8 (2006) 453-480.
12. Y. H. Ding, L. Jeanjean, Homoclinic orbits for a nonperiodic Hamiltonian system, *J. Differential Equations* 237 (2007) 473-490.
13. Y. H. Ding, C. Lee, Existence and exponential decay of homoclinics in a nonperiodic superquadratic Hamiltonian system, *J. Differential Equations* 246 (2009) 2829-2848.
14. W. X. Chen, M. B. Yang, Y. H. Ding, Homoclinic orbits of first order discrete Hamiltonian systems with super linear terms, *Science China Mathematics* vol. 54 No. 12 (2011) 2583-2596.
15. J. S. Yu, H. H. Bin, Z. M. Guo, Multiple periodic solutions for discrete Hamiltonian systems, *Nonlinear Anal.* 66 (2007) 1498-1512.
16. Z. M. Guo, J. S. Yu, Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems, *Nonlinear Anal.* 55 (2003) 969-983.
17. B. Zheng, Multiple Periodic Solutions to Nonlinear Discrete Hamiltonian Systems, *Advances in Difference Equations* Volume 2007.
18. X. Q. Deng, Periodic Solutions for Subquadratic Discrete Hamiltonian Systems, *Advances in Difference Equations* Volume 2007.
19. X. Q. Deng, G. Cheng, Homoclinic Orbits for Second Order Discrete Hamiltonian Systems, with Potential Changing Sign, *Acta. Appl. Math.* 103 (2008) 301-314.
20. C. D. Ahlbrandt, Equivalence of discrete Euler equations and discrete Hamiltonian systems, *J. Math. Anal. Appl.* 180 (1993) 498-517.
21. M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, *J. Math. Anal. Appl.* 199 (1996) 804-826.
22. S. Chen, Disconjugacy, disfocality, and oscillation of second order difference equations, *J. Differential Equations* 107 (1994) 383-394.
23. L. H. Erbe, P. Yan, Disconjugacy for linear Hamiltonian difference systems, *J. Math. Anal. Appl.* 167 (1992) 355-367.
24. P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, *Trans. Amer. Math. Soc.* 246 (1978) 1-30.